OFFICE OF NAVAL RESEARCH

CONTRACT N5 ORI-76 PROJECT ORDER X

NR - 384 - 903

TECHNICAL MEMORANDUM
NO. 25

PULSATIONS AND GROWTH OF
GAS-FILLED BUBBLES IN SOUND FIELDS

BY

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AUGUST 8, 1952

ACOUSTICS RESEARCH LABORATORY

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HARVARD UNIVERSITY-CAMBRIDGE, MASSACHUSETTS

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Abstract

The pulsations of a gas-filled bubble in a sound field
are discussed with particular attention to the effect of
the viscosity of the external liquid; the irreversible con-
duction of heat by the gas within the bubble; and the scat-
tering of energy by the bubble. The expressions obtained
for the gas pressure within the bubble and the time rate
of change of bubble radius are used to determine the rate
at which gas diffuses in and out of a bubble of mean radius
R. A threshold for bubble growth, called the threshold for
gaseous-type cavitation, is defined by the condition that
the net flow of gas across the surface of the bubble will
be zero. This average rate of gaseous diffusion can also
be used to determine in a stepwise fashion the mean change
of bubble radius with time. Transient effects are also
discussed.

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C O N T E N T S

PREFACE

This memorandum represents a continuation
of studies on the phenomenon of cavitation.
These studies were initiated at this Laboratory
by Dr. F. G. Blake, Jr. It is hoped that sev-
eral additional reports will be added to this
series. Thanks are especially expressed to
Professor F. V. Hunt, Director of the Acoustics
Research Laboratory, Harvard University, who
has guided much of this work.

NOTATION

$a \quad = \dfrac{\rho c_p}{K}$

$a_0 \quad =$ gas solubility constant

$a' \quad = \sqrt{\dfrac{a_2}{a_1}}$

$A \quad =$ amplitude of incident pressure wave

$\vec{A} \quad =$ vector potential

$b \quad = \dfrac{-1}{K}$

$c \quad =$ velocity of sound

$c_L \quad =$ velocity of sound for longitudinal wave

$c_p \quad =$ specific heat at constant pressure

$c_v \quad =$ specific heat of gas at constant volume

$c(r,t) \quad =$ gas concentration in liquid

$D \quad =$ gas diffusion constant

$g \quad = \dfrac{K_1}{K_2}$

$g_0 \quad =$ gas tension

$h_n(x) \quad = \sqrt{\dfrac{\pi}{2x}} \; H_{n+\frac{1}{2}}(x) =$ spherical Hankel function

$j_n(x) \quad = \sqrt{\dfrac{\pi}{2x}} \; J_{n+\frac{1}{2}}(x) =$ spherical Bessel function

$k \quad = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda}$

$K \quad =$ heat conductivity

$n \quad =$ moles

Notation (continued)

n_g = moles of gas

m' = $\cosh\phi - \cos\phi$

M = molecular weight

p = pressure

p_f = pressure term associated with viscosity

p_g = gas pressure within bubble

p_g' = variational gas pressure within bubble

p_s = scattered pressure wave

P_g' = complex amplitude of p_g'

P_i = average pressure within bubble = $P_o + \dfrac{2\sigma}{R}$

P_o = hydrostatic pressure

$P_n(y)$ = Legendre polynomial

q_s = radial particle velocity of scattered wave

Q = heat content

r = radial coordinate

R = radius of bubble

R_o = initial radius of bubble

R^* = universal gas constant

t = time

T_o = initial temperature of bubble and liquid

U_i = interval energy

v = volume

$\vec{V}$ = particle velocity vector

W = work

Notation (continued)

Z = complex impedance presented by bubble to incident sound wave = $U + iV$

γ = ratio of specific heats for gas = $\dfrac{c_p}{c_v}$

δ = $\sqrt{\dfrac{\omega a}{2}}$

η = $\dfrac{1}{2\sqrt{Dt}}$

ϑ = temperature

$\bar{\vartheta}$ = space average of temperature

λ = wavelength

μ = viscosity

ρ = density

σ = surface tension

ϕ = $\sqrt{2a\omega R}$

δ = scalar potential

ω = angular frequency

ω_r = resonant angular frequency of bubble

In general the subscripts "1" and "2" refer respectively to the regions internal and external to the bubble.

<u>Pulsations</u> <u>and</u> <u>Growth</u> <u>of</u>

<u>Gas-Filled</u> <u>Bubbles</u> <u>in</u> <u>Sound</u> <u>Fields</u>

by

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I

INTRODUCTION

This report presents a theoretical discussion of the manner
in which gas-filled bubbles in liquids react to the incidence of
a sound wave and then grow as a result of a net inward diffusion
of gas. This process, referred to as rectified diffusion, re-
sults when more gas diffuses into the bubble during its half-
cycle of expansion than diffuses out of the bubble during its
half-cycle of contraction. The theoretical formulation of this
problem, namely the diffusion of gas across a moving boundary,
results in a nonlinear equation. As a first approximation, the
solution to the diffusion equation is obtained for a bubble of
constant radius. In determining the rate of flow of gas across
the surface of the bubble the bubble radius (and area) is then
allowed to vary in accordance with the pressure variations with-
in the bubble. This method had been first suggested by Harvey
[1-6]* who considered the steady-state diffusion of gas into a
bubble of fixed radius. Harvey assumed that the liquid surround-
ing the bubble is nonviscous and the temperature within the bubble
constant. This work was later extended by Blake [7-10] who con-
sidered the periodic steady-state diffusion of gas into a bubble of
fixed radius. Blake assumed that the temperature within the bubble

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*References to articles included in the Bibliography are made
by numbers enclosed in brackets [ ].

is constant and that all of the energy within a sound wave inci-
dent upon the bubble is used to cause pressure variations within
the bubble. Thus, previous work with regard to the concept of
rectified diffusion had neglected the effect of the extinction
of energy by the bubble as a result of heat conduction within
it, scattering by it, and the resistive forces due to the vis-
cosity of the liquid. These effects are discussed in this
report. In addition, the transient solution of the diffusion
process, a factor which has not been examined by other research-
ers, is discussed. In order to obtain an expression that
describes the manner by which gas diffuses into the bubble,
relations will first be determined for the variations of tem-
perature within the bubble, and the pressure field external
and internal to the bubble.

II

TEMPERATURE VARIATIONS WITHIN THE BUBBLE

A. The Heat Conduction Equation

Let us assume that a gas-filled bubble is subject to a uni-
form pressure field such that the changes in pressure at internal
points are due to the fluctuations of its boundary. This assump-
tion holds true as long as the bubble's diameter is much less
than the wavelength of the incident sound wave. During the
oscillations of the bubble, gas will diffuse inward and outward,
but the amount of gas in the bubble will be essentially constant
over a single cycle at practically all frequencies of interest
in this report. Within the bubble, the first law of thermodynamics
is applicable to a small element of volume, v, subject to a pres-
sure, p. Thus,

$$\frac{dQ}{dt} = \frac{dU_i}{dt} + \frac{dw}{dt} = \frac{dU_i}{dt} + p\frac{dv}{dt} \qquad (2-1)$$

Schneider [11] has shown that the processes of evaporation and
condensation are sufficiently rapid in comparison with the

pulsation rate of the bubble so that the pressure of the vapor
within the bubble will be constant and determined solely by the
temperature of the liquid. In general the pressure of the gas
within the bubble will far exceed the pressure of the vapor.
According to the perfect gas law,

$$p_g \, dv + v dp_g = m_g \, R^* \, d\Theta_1, \qquad (2\text{-}2)$$

and assuming that the only heat conduction process takes place
in the gas

$$\frac{dQ}{dt} = Kv\nabla^2\Theta_1. \qquad (2\text{-}3)$$

Hence,

$$K_1\nabla^2\Theta_1 = \rho_1 c_{p_1} \frac{\partial\Theta_1}{\partial t} - \frac{dp_g}{dt}. \qquad (2\text{-}4)$$

Let us define

$$a = \frac{\rho c_p}{K}$$

and

$$b = \frac{-1}{K}.$$

Then in spherical coordinates for a spherically symmetric system,

$$\frac{\partial^2\Theta_1}{\partial r^2} + \frac{2}{r}\frac{\partial\Theta_1}{\partial r} - a_1\frac{\partial\Theta_1}{\partial t} - b_1\frac{dp_g}{dt} = 0, \qquad (2\text{-}5)$$

or

$$\frac{\partial^2 u_1}{\partial r^2} - a_1\frac{\partial u_1}{\partial t} - rb_1\frac{dp_g}{dt} = 0, \qquad (2\text{-}6)$$

where $u_1 = r\Theta_1.$

    The heat conduction equation external to the bubble is

$$\frac{\partial^2 u_2}{\partial r^2} - a_2\frac{\partial u_2}{\partial t} = 0. \qquad (2\text{-}7)$$

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* The single subscript, 1, will be used to denote the region
within the bubble; the single subscript, 2, will be used to de-
note the region external to the bubble.

The gas pressure within the bubble will be made up of an average pressure term P_1 and a variational part p'_g. We shall see in Chap. III that we can write

$$p'_g = P'_g \, e^{i\omega t} \, ,$$

and the coefficient P'_g may be complex.

We shall also show that

$$P_1 = P_0 + \frac{2\sigma}{R} \, .$$

B. Steady-State Solution

The steady-state solution of the two equations, (2-6) and (2-7), can be obtained by writing that

$$u = v(r)e^{i\omega t} \, .$$

Then

$$\frac{d^2 v_1}{dr^2} - i\omega a_1 v_1 - i\omega P'_g b_1 r = 0 \qquad\qquad (2\text{-}8)$$

and

$$\frac{d^2 v_2}{dr^2} - i\omega a_2 v_2 \qquad\qquad = 0 \, . \qquad\qquad (2\text{-}9)$$

As solutions of these equations, let us try

$$v_1 = A_1 \sinh q_1 r + B_1 \cosh q_1 r - c \qquad\qquad (2\text{-}10)$$

and

$$v_2 = A_2 \, e^{q_2 r} + B_2 \, e^{-q_2 r} - c_2 \, . \qquad\qquad (2\text{-}11)$$

Then, since $v_1 \longrightarrow 0$ as $r \longrightarrow 0$, and v_2 is finite as $r \longrightarrow \infty$,

$$v_1 = A_1 \sinh (1+i)\delta_1 r - \frac{b_1}{a_1} P'_g \, r \qquad\qquad (2\text{-}12)$$

and

$$v_2 = B_2 \, e^{-(1+i)\delta_2 r} \qquad\qquad (2\text{-}13)$$

where

$$\delta_1 = \sqrt[4]{\frac{\omega a_1}{2}} \quad \text{and} \quad \delta_2 = \sqrt[4]{\frac{\omega a_2}{2}}$$

The boundary conditions at the surface of the bubble based upon the continuity of temperature and heat flux are

$$\Theta_1\bigg]_{r=R} = \Theta_2\bigg]_{r=R} \quad \text{and} \quad K_1\frac{\partial\Theta_1}{\partial r}\bigg]_{r=R} = K_2\frac{\partial\Theta_2}{\partial r}\bigg]_{r=R} \qquad (2\text{-}14)$$

or

$$u_1\bigg]_{r=R} = u_2\bigg]_{r=R} \quad \text{and} \quad K_1\frac{\partial}{\partial r}\left(\frac{u_1}{r}\right)\bigg]_{r=R} = K_2\frac{\partial}{\partial r}\left(\frac{u_2}{r}\right)\bigg]_{r=R} \qquad (2\text{-}15)$$

By satisfying these boundary conditions we find that the temperature internal to the bubble is given by the expression

$$\frac{a_1}{b_1}(\Theta_1 - T_o) =$$

$$\left[\frac{[(1+i)\delta_2 R+1]\frac{R}{r}\sinh(1+i)\delta_1 r}{(1+i)[\frac{K_1}{K_2}\delta_1 R\cosh(1+i)\delta_1 R + \delta_2 R\,\sinh(1+i)\delta_1 R] + (1-\frac{K_1}{K_2})\sinh(1+i)\delta_1 R} - 1\right]$$

$$\cdot P_g^i\, e^{i\omega t} . \qquad (2\text{-}16)$$

This solution may be written as

$$\frac{a_1}{b_1}(\Theta_1 - T_o) = \left[\frac{R}{r}\left(\frac{u_o + iv_o}{m'}\right)(u_i^i + iv_1^i) - 1\right] P_g^i\, e^{i\omega t} , \qquad (2\text{-}17)$$

where

$$u_o = \sinh\delta_1 r \cos\delta_1 r \sinh\frac{\phi}{2}\cos\frac{\phi}{2} + \cosh\delta_1 r \sin\delta_1 r \cosh\frac{\phi}{2}\sin\frac{\phi}{2}$$

$$\bar{v}_o = \cosh\delta_1 r \sin\delta_1 r \sinh\frac{\phi}{2}\cos\frac{\phi}{2} - \sinh\delta_1 r \cos\delta_1 r \cosh\frac{\phi}{2}\sin\frac{\phi}{2}$$

$$u_1^i = \frac{u_1''}{w_1''}$$

$$v'_1 = \frac{v''_1}{w''_1}$$

$$u''_1 = \phi[a'(2-g+\frac{g\phi}{m'}\sinh\phi + a'\phi) + \frac{g}{m'}(\sinh\phi + \sin\phi) + \frac{2(1-g)}{\phi}]$$

$$v''_1 = \phi g[a'(\frac{\phi}{m'}\sin\phi - 1) - \frac{\sinh\phi - \sin\phi}{m'}]$$

$$w''_1 = (1-g+\frac{g\phi}{m'}\sinh\phi + a'\phi)^2 + (1-g+\frac{g\phi}{m'}\sin\phi)^2$$

$$m' = \cosh\phi - \cos\phi$$

$$\phi = \sqrt{2a_1\omega}\ R$$

$$g = \frac{K_1}{K_2}$$

$$a' = \sqrt{\frac{a_2}{a_1}}\ .$$

The ratio, $g = \frac{K_1}{K_2}$, for various liquids is as follows ($K_1 = 0.055 \times 10^{-3}$ cal/sec cm$^\circ$C for air):

Table 1

Liquid	$\frac{K_1}{K_2}$ (See Smith [12])
benzene	0.14
kerosene	0.15
castor oil	0.13
olive oil	0.14
water	0.04
carbon tetrachloride	0.20
acetone	0.13

When $r = R$, $u_0 = m'$, and $v_0 = 0$. Then if $g \ll 1$, the temperature at the surface of a bubble ($r = R$) is $\theta_1 - T_0 \approx 0$.

A plot of $u_1' - 1$ and v_1' as functions of ϕ when $r = R$ is presented in Fig. 1 and Table 2 for a given value of g and a'.

Table 2

ϕ	$u_1' - 1$	v_1
0.1	$- 0.07 \times 10^{-3}$	-0.16×10^{-3}
1.0	$- 2.00 \times 10^{-3}$	-2.13×10^{-3}
5.0	-10.40×10^{-3}	-2.55×10^{-3}
10.0	-11.80×10^{-3}	-1.37×10^{-3}
100.0	-13.00×10^{-3}	-0.14×10^{-3}

$$(g = 0.20, \quad a' = \sqrt{a_2/a_1} = 15)$$

Figure 1 represents the extreme case of carbon tetrachloride wherein the maximum change in temperature at the surface of the bubble would be expected. It is apparent that even if the peak pressure within the gas is as great as 10 atmospheres, the surface temperature will differ from the liquid temperature by $10.6^\circ C$ ($b_1/a_1 = -0.813 \times 10^{-4}$ for air).

By means of these curves and the values of u_0 and v_0 one can determine the temperature at various points within the bubble. For convenience one can let $\delta_1 r = n \frac{\phi}{2}$ where n is any positive fraction, $0 \leq n \leq 1$, such that $R/r = 1/n$.

The average value of Θ with respect to r will be

$$\bar{\Theta} = \frac{3}{R^3} \int_0^R r^2 \Theta \, dr. \tag{2-18}$$

By using the integral

$$\frac{3}{R^3} \int_0^R r^2 \frac{R}{r} \sinh sr \, dr = \frac{3}{sR} \cosh sR - \frac{3}{s^2 R^2} \sinh sR, \tag{2-19}$$

we obtain the expression

$$\frac{a_1}{b_1}(\bar{\theta} - T_o) = (\alpha_1 + i\beta_1)P'_g \, e^{i\omega t} \, , \qquad (2\text{-}20)$$

where

$$\alpha_1 = \frac{6}{\phi^2}\left\{ ga'(\cosh\phi - \cos\phi) + \frac{g}{2}a'\phi^2(\cosh\phi + \cos\phi)\right.$$

$$+ \sinh\phi[\phi a'(\frac{\phi}{2}a'+1-g)+1] - \sin\phi[\phi a'(\frac{\phi}{2}a'+1+g)+1]$$

$$\div [\frac{2}{\phi}(\frac{\phi}{2}a'+1-g)^2 + \frac{\phi}{2}a'^2](\cosh\phi - \cos\phi) + g^2\phi(\cosh\phi + \cos\phi)$$

$$\left. + 2g(\phi a'+1-g)\sinh\phi + 2g(1-g)\sin\phi \right\} - 1,$$

$$\beta_1 = \frac{6}{\phi^2}\left\{ [(\frac{\phi}{2}a'+1)(\frac{\phi}{2}a'+1-g)\frac{2}{\phi} + \frac{\phi}{2}a'^2](\cos\phi - \cos\phi)\right.$$

$$-g\phi(\frac{\phi}{2}a'+1)(\cosh\phi + \cos\phi)$$

$$+[2g(\frac{\phi}{2}a'+1) - (\frac{\phi^2}{2}a'^2 + \phi a' + 1)]\sinh\phi$$

$$+[2g(\frac{\phi}{2}a'+1) - (\frac{\phi^2}{2}a'^2 + \phi a' + 1)]\sin\phi$$

$$\div [\frac{2}{\phi}(\frac{\phi}{2}a'+1-g)^2 + \frac{\phi}{2}a'^2](\cosh\phi - \cos\phi) + g^2\phi(\cosh\phi + \cos\phi)$$

$$+ 2g(\phi a'+1-g)\sinh\phi + 2g(1-g)\sin\phi .$$

A plot of α_1 and β_1 is presented in Fig. 2 and Table 3, on the opposite page.

It is apparent from Fig. 2 that for $\phi < 3.0$ the pulsations of a bubble will be essentially isothermal; for $\phi > 30.0$ the pulsations of a bubble will be essentially adiabatic. In the transition region losses will occur as a result of the irreversible conduction of heat. We shall see later that the resonant radius of a bubble usually lies in the vicinity of this transition region.

$$-\rho c_p(\Theta_{1sfc} - T_o) = (u_i' - l + l v_i')p_g'$$

Θ_{1sfc} = surface temperature

T_o = initial temperature

p_g' = variational gas pressure within bubble

ϕ = $\sqrt{\dfrac{2\rho c_p \, \omega}{K}} \, R$

ρ = density of gas

c_p = specific heat of gas

R = radius of the bubble

ω = angular frequency

K = heat conductivity of gas

$\times 10^{-3}$

FIG. 1.   THE TEMPERATURE AT THE SURFACE OF A BUBBLE

$$-\rho c_p(\overline{\Theta}-T_o) = (\alpha_1 + i\beta_1)p_g'$$

$\overline{\Theta}$ = space average of temperature within gas-filled bubble

T_o = initial temperature

p_g' = variational gas pressure within bubble

$\phi = \sqrt{\dfrac{2\rho c_p \omega}{K}}\, R$

ρ = density of gas

c_p = specific heat of gas

R = radius of the bubble

ω = angular frequency

K = heat conductivity of gas

$\times 10^{-3}$

β_1

α_1

$\phi \longrightarrow$
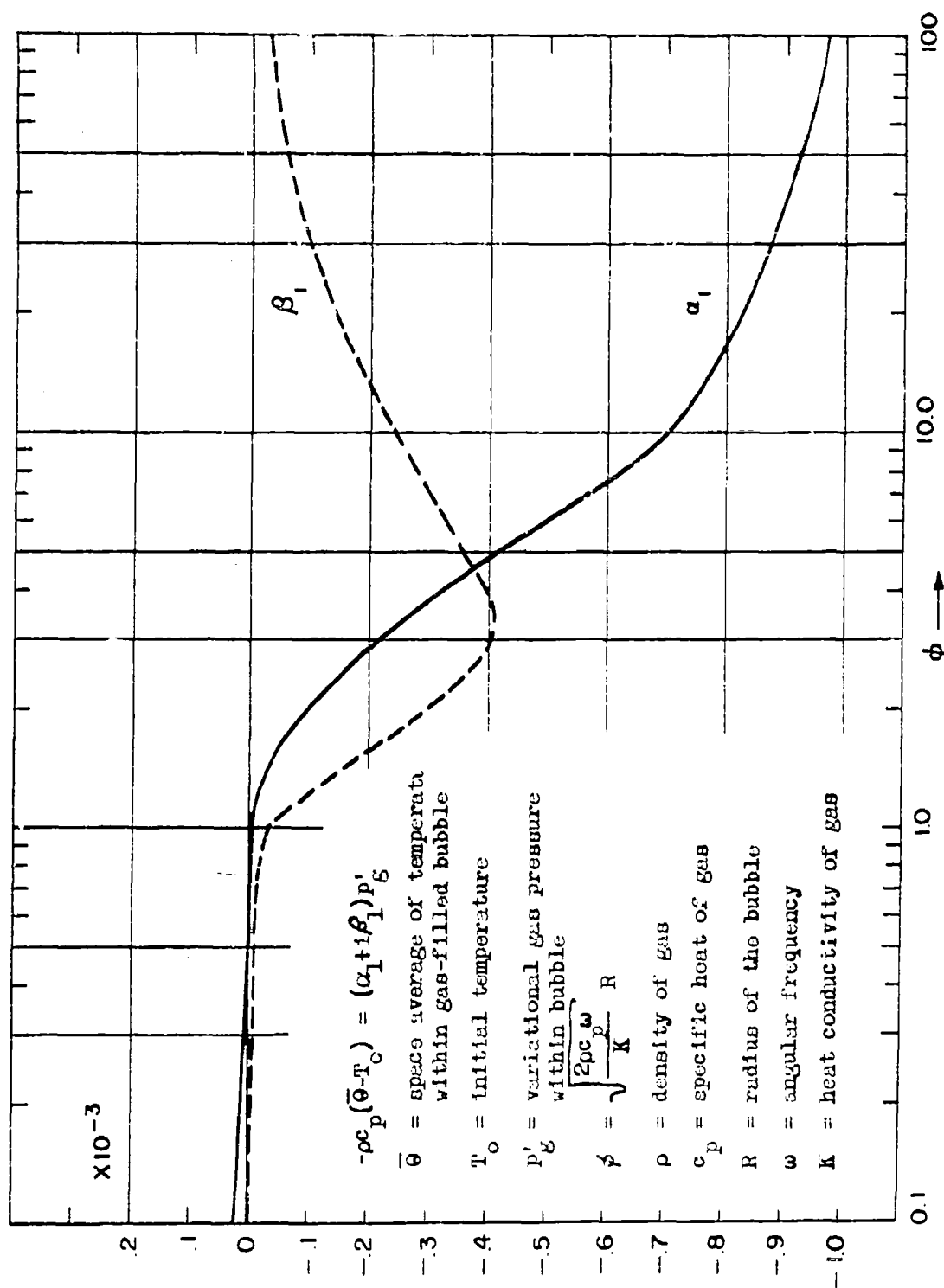
FIG. 2.   THE SPACE AVERAGE OF TEMPERATURE WITHIN A BUBBLE

Table 3

ϕ	$g = 0.04$ $a' = 12.2$		$g = 0.14$ $a' = 14.0$		$g = 0.20$ $a' = 15.0$	
	α_1	β_1	α_1	β_1	α_1	β_1
	$\times 10^{-3}$	$\times 10^{-3}$	$\times 10^{-3}$	$\times 10^{-3}$	$\times 10^{-3}$	$\times 10^{-3}$
0.1	23.3	$-$ 1.0	20.6	0	26.1	0
1.0	$-$ 2.2	$-$ 33.7	$-$ 3.1	$-$ 34.8	$-$ 3.8	$-$ 35.3
5.0	-391.8	-354.0	-395.5	-353.0	-397.3	-352.5
10.0	-701.0	-239.4	-702.9	-243.1	-703.4	-238.0
100.0	-970.1	$-$ 29.3	-970.3	$-$ 29.1	-970.4	$-$ 29.2

The perfect gas law for the bubble is

$$(P_1 + p_g') \frac{4}{3} \pi R^3 = m_g R^* \overline{\Theta}. \tag{2-21}$$

Now let R vary with p_g'. Then

$$\frac{dR}{dp_g'} = \left[\frac{mR^*}{\frac{4}{3}\pi R^3} \frac{d\overline{\Theta}}{dp_g'} - 1 \right] \frac{R}{3(P_1 + p_g')} \tag{2-22}$$

But by differentiating Eq. (2-20),

$$\frac{d\overline{\Theta}}{dp_g'} = \frac{b_1}{a_1} (\alpha_1 + i\beta_1)$$

and

$$\frac{dR}{R} = \left[\frac{mR^*}{V} \frac{b_1}{a_1}(\alpha_1 + i\beta_1) - 1 \right] \frac{dp_g'}{3(P_1 + p_g')}. \tag{2-23}$$

The quantities

$$\frac{b_1}{a_1} = -\frac{1}{\rho c_p}$$

$$V = \frac{mV}{\rho}$$

and

$$\frac{R^*}{M} = c_p - c_v .$$

Thus Eq. (2.23) becomes

$$\frac{dR}{R} = - \frac{\alpha - i\beta}{\gamma} \frac{dp_g}{3(P_1 + p_g)} , \qquad (2\text{-}24)$$

where

$$\gamma = \frac{c_p}{c_v}$$

$$\alpha = (\gamma - 1)\alpha_1 + \gamma$$

$$\beta = -(\gamma - 1)\beta_1$$

This equation expresses the relation between the change of the radius of the bubble and the pressure variations within the gas-filled bubble. This equation will be used later to study the pressure field external to the bubble.

C. Transient Solution

The complete solution (transient and steady-state terms) to the above problem can be obtained by means of Laplace transforms (see Carslaw and Jaeger [13], p. 288). We have seen that the temperature at the surface of a bubble will, in most cases, not differ greatly from the temperature of the surrounding liquid. Therefore, the importance of the transient term can be obtained from an analysis that assumes that the fluid reservoir in contact with the bubble is large enough and has a sufficiently high heat conductivity such that its temperature is a constant value T_0. At time $t = 0$, the temperature within the bubble is T_0, and the pressure within the bubble is $P_0 + \frac{2\sigma}{R}$. In addition, we shall momentarily assume that the radius of the bubble is constant (see Appendix I).

Let us define the Laplace transform of θ, namely,

$$\hat{\theta} = \int_0^\infty e^{-st} \theta(r,t)dt \qquad (2\text{-}25)$$

Equation (2-5) becomes

$$\frac{d^2\hat{\theta}_1}{dr^2} + \frac{2}{r}\frac{2\hat{\theta}_1}{dr} - a_1 s\hat{\theta}_1 + a_1\theta_1(t=o) - b_1\int_0^\infty e^{-st}\frac{dp_g}{dt}\,dt = 0 \quad (2\text{-}26)$$

or

$$\frac{d^2(r\hat{\theta}_1)}{dr^2} - a_1 sr\hat{\theta}_1 + a_1 r\theta_1(t=o) - b_1 r\int_0^\infty e^{-st}\frac{dp_g}{dt}\,dt = 0 \quad (2\text{-}27)$$

The initial and boundary conditions can then be written as

$$\theta(t=o) = T_o$$

$$\hat{\theta}(r,s) \text{ finite as } r \longrightarrow o$$

$$\hat{\theta}(R, s) = \frac{T_o}{s} \quad .$$

But since $p_g' = p_g - P_1$,

$$\int_0^\infty e^{-st}\frac{dp_g}{dt}\,dt = \int_0^\infty e^{-st}\frac{dp_g'}{dt} = s\hat{p}_g'\,.$$

Thus Eq. (2-27) becomes

$$\frac{d^2}{dr^2}(r\hat{\theta}_1) - a_1 s\eta\hat{\theta}_1 + a_1 rT_o - b_1 rs\hat{p}_g' = 0 \quad\quad (2\text{-}28)$$

As a solution of this equation let us try the substitution

$$r\hat{\theta}_1 = A\sinh qr + B\cosh qr + Cr \quad .$$

It follows that

$$q = \sqrt{a_1 s} \quad ,$$

and

$$C = \frac{T_o}{s} - \frac{b_1}{a_1}\hat{p}_g' \quad .$$

By virtue of the boundary conditions we obtain

$$B = 0$$

and

$$A = \frac{b_1}{a_1} \; \widehat{p_g'} \; \frac{R}{\sinh \sqrt{a_1 s} \; R}$$

Thus,

$$\widehat{\theta} = \frac{b_1}{a_1} \; \widehat{p_g'} \left[\frac{R \sinh \sqrt{a_1 s} \; r}{r \sinh \sqrt{a_1 s} \; R} - 1 \right] + \frac{T_0}{s} . \qquad (2\text{-}29)$$

The inverse transform of this equation is obtained by means of the inversion theorem

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i \infty}^{\gamma + i \infty} e^{\lambda t} \; \widehat{f(\lambda)} \; d\lambda . \qquad (2\text{-}30)$$

The term $\widehat{p_g'}$ is

$$\widehat{p_g'} = \frac{P_g'}{s - i\omega} .$$

By choosing the proper contour in the complex plane and by applying the theory of residues, it follows that

$$\frac{a_1}{b_1}(\theta_1 - T_0) = P_g' \; e^{i\omega t} \left[\frac{R \sinh \sqrt{i\omega a_1} \; r}{r \sinh \sqrt{i\omega a_1} \; R} - 1 \right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2\text{-}31)$$

$$+ P_g' \sum_{n=1}^{\infty} \frac{2n\pi(-1)^n e^{\frac{-n^2\pi^2 t}{aR^2}} \sin n\pi \frac{r}{R}}{aR^2 (\frac{n^2\pi^2}{aR^2} + i\omega)}$$

The average value of θ with respect to r is

$$\bar{\theta} = \frac{3}{R^3} \int_0^R r^2 \theta \; dr,$$

and the integrals

$$\frac{3}{R^3} \int_c^R r^2 \frac{R}{r} \sinh qr \; dr = \frac{3}{qR} \cosh qR - \frac{3}{q^2 R^2} \sinh qR$$

$$\frac{3}{R^3} \int_0^R r^2 \frac{R}{r} \sin n\pi \frac{r}{R} \; dr = \frac{3}{n\pi} (-1)^{n+1} \quad . \tag{2-32}$$

In the majority of cases the time constant $\frac{a_1 R^2}{2}$ will be at most of the order of magnitude of 10^{-4} seconds, and the transient term can be neglected.

Since

$$\frac{3}{\sqrt{i\omega a_1} \; R} \coth \sqrt{i\omega a_1} \; R = \frac{3}{\sqrt{2\omega a_1} \; R} \; [\sinh\sqrt{2\omega a_1} \; R - \sin\sqrt{2\omega a_1} \; R$$

$$- i(\sinh \sqrt{2\omega a_1} \; R + \sin \sqrt{2\omega a_1} \; R)], \tag{2-33}$$

we can write that

$$\frac{a_1}{b_1}(\bar{\theta} - T_0) = (\alpha_1' + i\beta_1') P_g' \; e^{i\omega t} \tag{2-34}$$

where

$$a_1' = \frac{3}{\phi} \frac{\sinh\phi - \sin\phi}{\cosh\phi - \cos\phi} - 1$$

$$\beta_1' = -\frac{3}{\phi} \frac{\sinh\phi + \sin\phi}{\cosh\phi - \cos\phi} + \frac{6}{\phi^2}$$

And as we would expect from Eq. (2-20), the terms

$$\left. \begin{array}{c} \alpha_1 \longrightarrow \alpha_1' \\[2mm] \beta_1 \longrightarrow \beta_1' \end{array} \right\} \quad \text{as } g \longrightarrow 0.$$

III

THE PRESSURE FIELD WITHIN THE BUBBLE

A. Exact Analysis

The acoustical field external and internal to a spherical gas bubble can be determined by an analysis similar to one developed by Epstein [14]. The small signal wave equation for a viscous liquid is

$$\rho \frac{\partial^2 \vec{v}}{\partial t^2} + \mu \nabla \times \nabla \times \frac{\partial \vec{v}}{\partial t} - \frac{4}{3}\mu \nabla (\nabla \cdot \frac{\partial \vec{v}}{\partial t}) - \rho c_L^2 \nabla (\nabla \cdot \vec{v}) = 0 \quad (3\text{-}1)$$

where

$$c_L = (\frac{dp}{d\rho})^{\frac{1}{2}} .$$

By writing in terms of a scalar and vector potential

$$\vec{v} = -\nabla \delta + \nabla \times \vec{A}, \quad (3\text{-}2)$$

and assuming periodic time dependence, it follows that

$$\nabla^2 \delta + k^2 \delta = 0, \qquad \nabla^2 \vec{A} + \bar{k}^2 \vec{A} = 0 , \quad (3\text{-}3)$$

where

$$k^2 = \omega^2 [c_L^2 + \frac{4}{3} i \frac{\omega \mu}{\rho}]^{-1} , \qquad \bar{k}^2 = -i\omega \frac{\rho}{\mu} .$$

Consider the case of a purely longitudinal primary wave, $A \sin \omega t (\text{Im } Ae^{i\omega t})$, that is incident upon a gas bubble in a viscous liquid. If the origin of a spherical coordinate system is taken at the center of the bubble with the polar axis in the direction of the incident wave, axial symmetry will require that $A_r = A_\theta = 0$. Both δ and A_ϕ can be expanded into the following series:

$$\delta_{i_2} = e^{ikr\cos\theta} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(k_2 r) P_n(\cos\theta)$$

$$\bar{\Phi}_{r_2} = \sum_{n=o}^{\infty} i^n(2n+1)B_{r_2} \, h_n(k_2 r)P_n(\cos\Theta)$$

$$A_{\phi_{r_2}} = \sum_{n=o}^{\infty} i^n(2n+1)C_{n_2}h_n(\bar{k}_2 r)\frac{d}{d\Theta} P_n(\cos\Theta)$$

$$\bar{\Phi}_1 = \sum_{n=o}^{\infty} i^n(2n+1)B_{n_1} j_n(k_1 r)P_n(\cos\Theta)$$

where the subscript "i" refers to the incident wave; the subscript
"r" refers to the reflected wave; the number "2" refers to the ex-
terior of the bubble; the number "1" refers to the interior of
the bubble; and

$$j_n(x) = \sqrt{\frac{\pi}{2x}} \; J_{n+\frac{1}{2}}(x) = \text{spherical Bessel function}$$

$$h_n(x) = \sqrt{\frac{\pi}{2x}} \; H_{n+\frac{1}{2}}(x) = \text{spherical Hankel function}$$

$$P_n(y) = \text{Legendre polynomial.}$$

Let $k_2 R = c_2$, $\bar{k}_2 R = d_2$ and $k_1 R = c_1$. Then for the liquids and
bubble sizes with which we shall be concerned, $c_1^2 \ll 1$, $c_2^2 \ll 1$,
$k_2 \approx \frac{\omega}{c_{L_2}}$ and $k_1 \approx \frac{\omega}{c_{L_1}}$.

The boundary conditions are such that the components of $\vec{v}$ as
well as the stresses must be continuous at the surface of the
bubble with due allowance for capillary forces. The expansions
for $j_n(x)$ and $h_n(x)$ are:

$$j_n(x) = \frac{x^n}{1.3 \ldots (2n+1)}\left[1 - \frac{x^2}{2(2n+3)} + \frac{x^4}{2.4(2n+3)(2n+5)} \ldots \right] n \geq 0,$$

$$j_{-n-1}(x) = (-1)^n 1.3 \ldots (2n+1)x^{-n-1}\left[1 + \frac{x^2}{2(2n-1)} + \frac{x^4}{2.4(2n-1)(2n-3)} + \ldots \right]$$

(3-5)

$$n \geq 1$$

$$j_{-1}(x) = \frac{1}{x} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right]$$

and

$$h_n(x) = j_n(x) - i(-1)^n j_{-n-1}(x) .$$

For the first term of the series, that is, $n = 0$, one can show by using these expansions for $j_n(x)$ and $h_n(x)$ that

$$\frac{3iB_{o_2}}{c_2} + B_{o_1} c_1^2 = c_2^2$$

$$(d_2^2 - 4)i \frac{B_{o_2}}{c_2} + B_{o_1} \frac{d_2^2}{f} Y = d_2^2, \qquad (3-6)$$

where

$$Y = 1 - \frac{2\sigma}{R} \frac{\alpha - i\beta}{3\gamma P_1} , \qquad f = \frac{\rho_2}{\rho_1}$$

and it has been assumed that

$$p_g^i \ll P_1 .$$

Elimination of B_{o_2} between these two equations leads to

$$B_{o_1} \approx \left[\frac{Y}{f} - \frac{1}{3}(1 - \frac{4}{d_2^2})c_1^2 \right]^{-1} . \qquad (3-7)$$

Hence the first term of the series for $\bar{s}_1$ is

$$B_{o_1} j_{o_1}(k_1 r) \approx \frac{\rho_2 c_{L_1}^2}{\rho_1 c_{L_1}^2 \ Y - \frac{\rho_2 \omega^2 R^2}{3} + i \frac{4\omega\mu}{3}} . \qquad (3-8)$$

But

$$\rho_1 c_{L_1}^2 \approx \frac{\gamma P_1}{\alpha - i\beta} \qquad (3-9)$$

and

$$p_g^i \approx \rho_1 B_{o_1} \ j_o(k_1 r) \frac{A}{\rho_2} \ e^{i\omega t} . \qquad (3-10)$$

Thus,

$$p_g' \approx \frac{\gamma \, P_{1/\alpha - i\beta}}{\frac{\gamma P_1 \gamma}{\alpha - i\beta} - \frac{\rho \omega^2 R^2}{3} + i \, \frac{4\omega\mu}{3}} \, A \, e^{i\omega t} \qquad (3\text{-}11)$$

The coefficient B_{1_1} can be determined by taking the boundary condition for $n=1$, subject to the same approximations used in obtaining B_{0_1}. One can easily show that

$$B_{1_1} \approx \frac{c_2}{c_1} \, \frac{7 + d_2^2}{7 - d_2^2} \, . \qquad (3\text{-}12)$$

The additional term due to B_{1_1} in the series expansion for δ_1 is

$$i \, \frac{\omega}{c_{L_2}} \, \frac{7 + d_2^2}{7 - d_2^2} \, r \, .$$

Denote the correction to p_g' due to this additional term as p_g''. Then

$$\frac{p_g'' \Big]_{r=R}}{p_g' \Big]_{r=R}} \approx \left(-\frac{4}{3} \, \frac{\omega^2}{c_{L_1}^2} \, \frac{R}{\rho c_{L_2}} + i \, \frac{\omega R}{f \, c_{L_2}} \right) \ll 1. \qquad (3\text{-}13)$$

Thus if $kR \ll 1$, the pressure distribution over the surface of the bubble will be uniform. For bubbles whose size becomes comparable to a wavelength, the additional terms of the series cannot be neglected.

B. Approximate Analysis

Equation (3-11) for p_g' can be obtained in far more simple manner [15] if one makes the additional assumption that the terms pertaining to viscosity in the wave equation are negligible. Then the scalar potential for spherically symmetric pulsations in the liquid is simply

$$\delta = \delta_0 \, \frac{e^{i(\omega t - kr)}}{r} \, . \qquad (3\text{-}14)$$

Thus

$$p_s = \rho \frac{\partial \bar{\phi}}{\partial t} \qquad (3\text{-}15)$$

and

$$q_s = - \frac{\partial \bar{\phi}}{\partial r} \qquad (3\text{-}15)$$

where

p_s = pressure in radially scattered sound wave,

q_s = particle velocity in radially scattered sound wave.

Up to this point the effect of the viscosity of the fluid has been neglected. As a result of viscosity an additional pressure term, p_f, will appear. The approximate value of p_f can be obtained in the following manner. The pressure tensor in a liquid of viscosity, μ, is, for the case of radial symmetry,

$$p_f = \frac{2\mu}{3r^2} \frac{\partial}{\partial r}(r^2 q_s) - 2\mu \frac{\partial q_s}{\partial r} . \qquad (3\text{-}16)$$

The pulsation of the bubble will be in accordance with the thermodynamic equation (2-24),

$$\frac{dV}{V} = \frac{3}{R} dR = - \frac{\alpha-\beta}{\gamma} \frac{dp_g^i}{P_i} ,$$

where it has been assumed that

$$p_g^i \ll P_i .$$

At the surface of the bubble the continuity of pressure and velocity requires that

$$P_i + p_g = Ae^{i\omega t} + p_s \Big)_{r=R} + p_f \Big)_{r=R} + \frac{2\sigma}{R}$$

or

$$\frac{d}{dt} p_g^i = \frac{d}{dt} \left(A e^{i\omega t} + p_s + p_f \right)_{r=R} - \frac{2\sigma}{R^2} \frac{dR}{dt} \qquad (3\text{-}17)$$

and

$$\frac{dR}{dt} = q_s \bigg)_{r=R} \tag{3-18}$$

The average value, P_i, with respect to time, is simply $(P_o + \frac{2\sigma}{R})$, where R is a mean value. Thus we can obtain the expression for p_g', namely,

$$p_g' \approx \frac{3\gamma \ P_i/\alpha-i\beta}{\dfrac{3\gamma \ P_i \gamma}{\alpha-i\beta} - \dfrac{\omega^2 R^2}{1+k^2 R^2} + i4\omega\mu + i \dfrac{\omega^3 R^3}{c(1+k^2 R^2)}} \ \hat{A} \ e^{i\omega t} \tag{3-19}$$

which is the same as Eq. (3-11) for $kR \ll 1$.

C. Impedance of Bubble and Resonance

In a similar manner we can determine the impedance presented by the bubble to a sound wave, namely,

$$Z = \frac{\hat{A} \ e^{i\omega t}}{(q_s)_{r=R}}$$

$$= U + iV \tag{3-20}$$

where

$$U = \frac{4\mu}{R} + \frac{\rho c k^2 R^2}{1+k^2 R^2} + \frac{3\gamma \ P_i \beta}{\omega R(\alpha^2+\beta^2)}$$

$$V = \frac{\omega\rho R}{1+k^2 R^2} - \frac{1}{\dfrac{\omega R(\alpha^2+\beta^2)}{3\gamma P_i \alpha - \dfrac{2\sigma}{R}(\alpha^2+\beta^2)}} \ .$$

The terms in U result, respectively, from (1) the work accomplished by the incident pressure in overcoming resistive forces due to the viscosity of the liquid; (2) the energy scattered by the bubble; (3) the energy lost as a result of the irreversible conduction of heat within the bubble. The terms in V respectively result from (1) the inertial reaction upon the bubble of the entrained fluid that moves with the surface of the bubble; (2) the compressibility of the gas within the bubble.

A plot of the terms U and V as functions of ϕ at a frequency

of 60 kc/s is presented in Fig. 3 and Table 4, below. Two
liquids have been considered, namely, water and olive oil. The
bubble is assumed to be air-filled.

Table 4

		Water		Olive Oil	
β	R(cm)	U	V	U	V
0.02	1×10^{-5}	400.0	-8.52×10^6	336000	-4.31×10^6
0.1	5.1×10^{-5}	790.5	-4.59×10^5	66403.2	-2.94×10^5
1	5.1×10^{-4}	79.3	-1.86×10^4	6640.3	-1.71×10^4
5	2.5×10^{-3}	64.2	-2.68×10^3	1374.7	-2.69×10^3
10	5.1×10^{-3}	50.4	$- 85.7$	704.6	$- 216.9$
100	5.1×10^{-2}	2530	$+1.89 \times 10^4$	2406	$+1.73 \times 10^4$

The steady-state response (or impedance) of the bubble in-
dicates to some extent the appearance of its transient pulsa-
tions. The impedance consists of resistive, inertial, and
stiffness terms. Let us assume that these terms, to a first
approximation, are constant. For small values of R, the reac-
tance will be negative -- that is, the bubble will be stiffness-
controlled. Under these conditions one can show [16] that it
is possible for an underdamped or oscillatory condition to
exist in "non-viscous" liquids wherein a high-frequency transient
of rather large magnitude will be superimposed on the steady-state
pulsations if the applied signal has the proper phase. In other
words, under special conditions the transient pulsations of a
bubble could be important in affecting the initial diffusion of
gas inward and outward. We shall not, however, discuss this
problem in this memorandum.

Equation (3-19) can now be written

$$p'_g = \frac{3 \gamma P_1/(a-1\beta)}{i\omega RZ} \ A \ e^{i\omega t} = p'_g \ e^{i\omega} \qquad (3-21)$$

$$\phi = \sqrt{\frac{2\rho c_p \omega}{K}} \; R$$

WATER (NEGATIVE IMAGINARY PART)

OLIVE OIL (NEGATIVE IMAGINARY PART)

OLIVE OIL (REAL PART)

WATER (POSITIVE IMAGINARY PART)

OLIVE OIL (POSITIVE IMAGINARY PART)
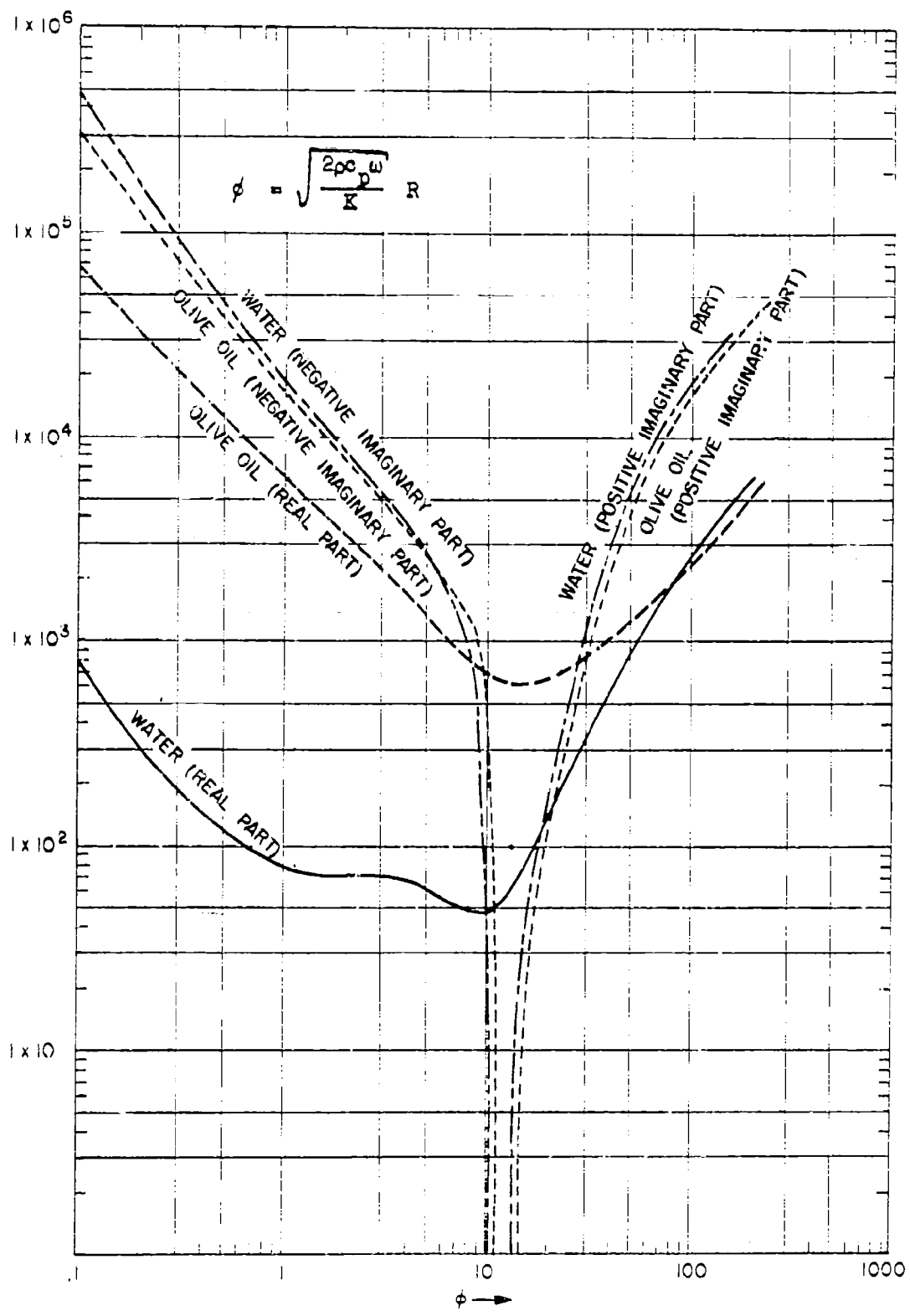
WATER (REAL PART)

$\phi \longrightarrow$

FIG. 3.   THE COMPLEX IMPEDANCE PRESENTED BY A BUBBLE TO A SOUND WAVE

Resonant pulsations of the bubble will occur when $V = 0$, or

$$\omega_r^2 = \frac{\omega_1^2}{1-(\omega_i^2/c^2)R^2} \,, \tag{3-22}$$

where

$$\omega_1^2 = \omega_0^2 \frac{1 + \frac{2\sigma}{RP_0}(\, - \, \frac{\alpha+\beta^2/\alpha}{3\gamma})}{\alpha + \frac{\beta^2}{\alpha}} \,, \qquad \omega_0^2 = \frac{1}{R^2} \frac{3\gamma P_0}{\rho}$$

Similarly, we can define a resonant bubble size $R = R_r$, when $\omega = \omega_r$. Thus,

$$Z = U + i \frac{\omega\rho R}{1+k^2R^2}(1 - \frac{\omega_r^2}{\omega^2}) \,, \tag{3-23}$$

and the displacement of the surface of a bubble is, respectively,

$$\left. \xi_s \right)_{r=R} = \frac{A \, e^{i\omega t}}{\omega\left(\frac{4\mu}{R} - i \, \frac{3(P_0 + \frac{2\sigma}{R})}{R}\right)} \qquad \text{for} \qquad R \ll R_r$$

$$\left. \xi_s \right)_{r=R} = \frac{A \, e^{i\omega t}}{\omega U} \qquad \text{for} \qquad R = R_r$$

$$\left. \xi_s \right)_{r=R} = \frac{A \, e^{i\omega t}}{\omega\left(\frac{\rho c k^2 R^2 + i\omega\rho R}{1 + k^2R^2}\right)} \qquad \text{for} \qquad R \gg R_r$$

Similarly,

$$p_g' = \frac{3(P_0 + \frac{2\sigma}{R})}{iR} \left. \xi_s \right)_{r=R} \qquad \text{for} \qquad R \ll R_r$$

$$p_g' = \frac{3\gamma(P_0 + \frac{2\sigma}{R})/\alpha-i\beta}{iR} \left. \xi_s \right)_{r=R} \qquad \text{for} \qquad R = R_r$$

$$p_g^i = \frac{3\gamma(P_0 + \frac{2\sigma}{R})}{iR} \left. \xi_s \right)_{r=R} \qquad \text{for} \qquad R \gg R_r$$

The strain set up at the surface of the bubble is $\dfrac{\chi_s)_{r=R}}{R}$.
This strain will be very large. The corresponding
pressure will not be very large. However, if any material is
near or in contact with the bubble, it may be mechanically
damaged as a result of repeated deformations associated with
the large strains. Some materials will effectively change
the character of the pulsations of the bubble such that the
above analysis will not hold true. And, finally, the effects
of numerous bubbles are extremely complicated and can only be
estimated on the basis of the above analysis.

IV

THE RECTIFIED DIFFUSION OF GAS

A. Complete Solution to the Diffusion Equation

The diffusion of gas into a bubble will be in accordance
with the diffusion equation

$$\nabla^2 c = \frac{1}{D} \frac{\partial c}{\partial t} . \qquad (4-1)$$

For a moving boundary-value problem this equation is nonlinear
(see Appendix I). We shall assume that the radius of the bub-
ble is momentarily constant so that an exact solution to this
equation can be obtained. This assumption means that the dif-
fusion process occurs as a result of variations of the gas
pressure within the bubble. These variations in pressure result
from the continuity of pressure across the surface of the bubble.
We shall later make an approximation to account for the move-
ments of the bubble's surface.

During the positive half-cycle gas will diffuse into the
liquid which is undersaturated with respect to the bubble; during
the negative half-cycle gas will diffuse into the bubble which
is undersaturated with respect to the liquid. But the surface
area of the bubble will be greater during the negative half-
cycle such that there will be a net sonically induced diffusion

of gas into the bubble. If the rate of this net influx exceeds
the rate at which gas dissolves as a result of the internal ex-
cess pressure due to surface tension, the bubble will grow.
This process, called "rectified diffusion," can be studied by
means of the diffusion equation.

In spherical coordinates for a spherically symmetric
system the diffusion equation is

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{D} \frac{\partial u}{\partial t} \qquad\qquad (4\text{-}2)$$

where

$$u = \frac{rc}{a_0} \qquad .$$

The boundary and initial conditions are

$$c(r,t) \longrightarrow a_0 g_0 \quad\text{as}\quad r \longrightarrow \infty$$

$$c(R,t) = a_0 p_g \qquad\qquad (4\text{-}3)$$

$$c(r,o) = a_0 g_0 + \frac{R}{r} a_0 (p_g(t=o)-g_0),$$

or

$$u(r,t) \longrightarrow r g_0 \quad\text{as}\quad r \longrightarrow \infty$$

$$u(R,t) = R p_g = R(P_1 + a_1' A \sin(\omega t + X')) = W_1(t)$$

$$u(r,o) = r g_0 + R(P_1 - g_0) + R a_1' A \sin X' = W_2(r)$$
$$(4\text{-}4)$$

where

$$a_1' = -\frac{3\gamma P_1}{\sqrt{\alpha^2 + \beta^2}} a_1 \quad,$$

$$a_1 = \frac{1}{\omega R \sqrt{U^2 + V^2}} \quad,$$

$$X' = h + \ell \quad,$$

$$h = \tan^{-1} \frac{U}{V} \quad.$$

$$\ell = \tan^{-1} \frac{\beta}{\alpha}$$

The complete solution of the diffusion equation in the infinite region bounded by the internal sphere of radius R is readily obtainable by the use of Fourier integrals and Laplace transforms. A complete discussion of these techniques is given in the textbooks of Carslaw and Jaeger [13] and Churchill [17]. The solution is (see Carslaw and Jaeger [13], p. 209)

$$u = \frac{1}{2\sqrt{\pi D t}} \int_R^\infty W_2(\lambda) \left[e^{-\frac{(r-\lambda)^2}{4Dt}} - e^{-\frac{(r+\lambda-2R)^2}{4Dt}} \right] d\lambda$$

$$\tag{4-5}$$

$$+ \frac{2}{\sqrt{\pi}} \int_{\frac{r-R}{2\sqrt{Dt}}}^\infty W_1\left(t - \frac{(r-R)^2}{4Dy^2} \right) e^{-y^2} dy \ .$$

Let $\eta = \frac{1}{2\sqrt{Dt}}$, $z = (\lambda-r)\eta$, and $z' = (r+\lambda-2R)\eta$. Then,

$$u = \frac{1}{\sqrt{\pi}} \int_{(r-R)\eta}^\infty W_2\left(\frac{z}{\eta} + r\right) e^{-z^2} dz - \frac{1}{\sqrt{\pi}} \int_{(r-R)\eta}^\infty W_2\left(\frac{z}{\eta} -r+2R\right) e^{-z^2} dz$$

$$\tag{4-6}$$

$$+ \frac{2}{\sqrt{\pi}} \int_{(r-R)\eta}^\infty W_1\left(t - \frac{(r-R)^2 \eta^2 t}{z^2} \right) e^{-z^2} dz \ ,$$

where the integration parameter is now denoted by z. Suppose that the boundary conditions are those of Eq. (4-4). Then the solution of the diffusion equation is

$$u(r,t) = rg_0 + R(P_1 - g_0) + Ra_1' A \sin X' \ \mathrm{erf} \frac{r-R}{2\sqrt{Dt}}$$

$$+ Ra_1' A \ e^{-(r-R)\sqrt{\frac{\omega}{2D}}} \sin\left(\omega t + X' - (r-R)\sqrt{\frac{\omega}{2D}} \right)$$

$$+ 2Ra_1' A \cos X' \frac{\omega D}{\pi} \int_0^\infty \frac{z e^{-Dtz^2}}{\omega^2 + D^2 z^4} \sin z(r-R) dz$$

$$- 2Ra_1'A \sin X' \frac{D^2}{\pi} \int_0^\infty \frac{z^3 e^{-Dtz^2}}{\omega^2 + D^2 z^4} \sin z(r-R) dz. \qquad (4-7)$$

Thus, for $t > 0$,

$$\left.\frac{\partial c}{\partial r}\right)_{r=R} = a_o\left(-\frac{u}{r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)_{r=R}$$

$$= -\frac{a_o}{R}\left[P_1 - g_o + a_1'A \sin(\omega t + X') - \frac{Ra_1'A \sin X'}{\sqrt{\pi D t}}\right]$$

$$- a_o a_1'A \sqrt{\frac{\omega}{2D}} \left[\sin(\omega t + X') + \cos(\omega t + X')\right]$$

$$+ 2a_o a_1'A \cos X' \frac{\omega}{\pi D} \int_0^\infty \frac{z^2 e^{-Dtz^2}}{\frac{\omega^2}{D^2} + z^4} dz$$

$$- 2a_o a_1'A \sin X' \frac{1}{\pi} \int_0^\infty \frac{z^4 e^{-Dtz^2}}{\frac{\omega^2}{D^2} + z^4} dz \qquad (4-8)$$

The term

$$R_n = \int_0^\infty \frac{z^{2n} e^{-Dtz^2}}{\frac{\omega^2}{D^2} + z^4} dz, \qquad t > 0 \qquad (4-9)$$

has, to this author's knowledge, not previously been evaluated in closed form. The evaluation will therefore be given in considerable detail. Quite often, integrals of the form

$$\int_0^\infty \phi(z) e^{-az^2} dz$$

cannot be solved by Cauchy's method of contour integration in the complex plane. The most common example is the error function of infinity, namely, the definite integral

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} .$$

Other techniques in the real plane must be used to determine these integrals. The integral R_n can be written:

$$R_n = \int_0^\infty \frac{z^{2n} e^{-Dtz^2}}{\frac{\omega^2}{D^2} + z^4} dz = \frac{(-1)^n}{D^n} \frac{d^n}{dt^n} R_0 \qquad (4\text{-}10)$$

But,

$$R_0 = \int_0^\infty \frac{e^{-Dtz^2}}{\frac{\omega^2}{D^2} + z^4} dz = \frac{D\sqrt{Dt}}{2i\omega} \int_0^\infty e^{-z^2} \left[\frac{1}{z^2 - i\omega t} - \frac{1}{z^2 + i\omega t} \right] dz$$

$$= \frac{D\sqrt{Dt}}{2i\omega} \int_0^\infty e^{-z^2} dz \int_0^\infty \left[e^{-(z^2 - i\omega t)x} - e^{-(z^2 + i\omega t)x} \right] dx$$

$$= \frac{D}{\omega}\sqrt{Dt} \int_0^\infty \sin(\omega t x) dx \int_0^\infty e^{-(1+x)z^2} dz$$

$$= \frac{D}{2\omega} \sqrt{\pi Dt} \int_0^\infty \frac{\sin(\omega t x)}{(1+x)^{\frac{1}{2}}} dx$$

$$= \frac{\pi D}{2\omega} \sqrt{\frac{2D}{\omega}} [(\tfrac{1}{2} - S(\omega t))\cos\omega t - (\tfrac{1}{2} - C(\omega t))\sin\omega t], \qquad (4\text{-}11)$$

where $S(\omega t)$ and $C(\omega t)$ are the Fresnel integrals defined by

$$S(\omega t) = \int_0^{\omega t} \frac{\sin x}{\sqrt{2\pi x}} dx$$

$$C(\omega t) = \int_0^{\omega t} \frac{\cos x}{\sqrt{2\pi x}} dx .$$

Thus,

$$R_1 = \frac{\pi}{2} \sqrt{\frac{2D}{\omega}} \, [(\tfrac{1}{2} - S(\omega t))\sin\omega t + (\tfrac{1}{2} - C(\omega t))\cos\omega t]$$

$$(4-12)$$

Similar techniques can be used to evaluate many integrals of the forms

$$\int_0^\infty \phi(z) e^{-az^2} \, dz.$$

The associated trigonometric forms of these integrals may also be evaluated by means of integration in the complex plane.

B. Rectified Diffusion

Since the rate at which gas flows into a bubble in moles/sec is

$$\dot{m} = 4\pi R^2 D(\frac{\partial \phi}{\partial r})_{r=R} \qquad (4-13)$$

we can now determine the average value of $\dot{m}$ over one cycle. However, let us now introduce the variation $R(t)$.

The radius $R(t)$ of the bubble can be expanded in a Taylor's series about its mean or initial value, R_0. This series is

$$R = R_0 + \frac{dR}{dp_g}\bigg]_0 \Delta p_g + \ldots \, ,$$

where

$$p_g = P_i + p_g^i$$

or

$$\Delta p_g = p_g^i.$$

Substitution of Eqs. (2-24) and (3-21) leads to the series

$$R = R_0\left(1 + \frac{A}{\omega R_0 \sqrt{U^2 + V^2}} \sin(\omega t + \tan^{-1} \frac{U}{V}) + \ldots \right)$$

$$= R_o(1 + a_1 A \sin(\omega t + h) + \ldots)$$ (4-14)

Then

$$\overline{\dot{m}} = -4\pi R_o a_o D \left[P_1 - g_o + \frac{a_1 a_1'}{2}(\cos\ell - \sin\ell)(1+2R_o\sqrt{\frac{\omega}{2D}})A^2 \right]$$ (4-15)

where $\ell = \tan^{-1}\beta/\alpha$,

since it can be shown graphically that

$$\overline{S(\omega t)\sin\omega t} \approx \overline{C(\omega t)\cos\omega t} \approx \overline{C(\omega t)\cos\omega t \sin\omega t}$$
$$\approx \overline{C(\omega t)\sin\omega t} \approx \overline{C(\omega t)\cos\omega t \sin^2\omega t}$$
$$\approx \overline{C(\omega t)\sin^3\omega t} \approx \overline{S(\omega t)\cos\omega t}$$
$$\approx \overline{S(\omega t)\cos\omega t \sin\omega t}$$
$$\approx \overline{S(\omega t)\cos\omega t \sin^2\omega t} \approx \overline{S(\omega t)\sin^3\omega t}$$
$$\approx 0$$ (4-16)

and

$$\overline{S(\omega t)\sin^2\omega t} \approx 0.25$$
$$\overline{C(\omega t)\sin^2\omega t} \approx 0.25$$ (4-17)

The terms in Eq. (4-16) represent the effect of the transient terms in the solution of the diffusion equation. Even if the graphical averaging of these terms is taken over the first few cycles of the applied sound signal, the average of these terms will be approximately zero. Thus the transient part of the solution has a negligible effect upon the average rate at which gas moles diffuse across the surface of the bubble.

In order to have (on the average) no moles of gas entering or leaving the bubble, the sound amplitude, A, must be large enough such that $\overline{\dot{m}} = 0$. Let us denote the sound amplitude that satisfies this condition as A_∞, the threshold for the growth of gas-filled bubbles (also referred to as the threshold for gaseous-type cavitation). Then,

$$A_\infty = \sqrt{\frac{2(P_o - g_o + \frac{2\sigma}{R_o})}{-a_1 a_1'(\cos\ell - \sin\ell)(1 + 2R_o\sqrt{\frac{\omega}{2D}})}}$$

$$= \sqrt{\frac{2(P_o + \frac{2\sigma}{R_o} - g_o)\,\omega^2 R_o^2(U^2 + V^2)\sqrt{\alpha^2 + \beta^2}}{3\gamma(P_o + \frac{2\sigma}{R_o})(\cos\ell - \sin\ell)(1 + 2R_o\sqrt{\frac{\omega}{2D}})}}$$

$$(4\text{-}18)$$

A plot of A_∞ as a function of ϕ is presented in Fig. 4 and Table 5. The frequency chosen is 60 kc/s, and two liquids, water and olive oil, have been considered. The bubble is assumed to be air-filled.

Table 5

ϕ	R(cm)	**Water** A_∞ (Atmospheres)	**Olive Oil** A_∞ (Atmospheres)
0.02	1×10^{-5}	14.8	4.7
0.1	5.1×10^{-5}	1.9	0.64
1	5.1×10^{-4}	0.14	0.06
5	2.5×10^{-3}	0.02	0.01
10	5.1×10^{-3}	0.008	0.002
100	5.1×10^{-2}	0.13	0.05

C. Stepwise Growth of Bubble

For a gas bubble nucleus of radius R_o, much smaller than resonant size, a sound amplitude A greater than A_∞ will be required for its growth. As the bubble becomes larger the sound amplitude required for its growth (A_∞^+) becomes less and less. If the initial sound amplitude, A_∞^+, is maintained constant, the rate of growth of the bubble will be more and more rapid as the bubble increases in size. However, because of the

additional losses that occur when a bubble approaches its
resonant size, the rate of growth will be retarded for the
same A_∞^+, and the pulsations of the bubble will become very
complex. For bubbles greater than resonant size, the rate
of growth will decrease with increasing bubble radius. Hence,
we can see qualitatively that the mean radius of the bubble
should follow an S-curve with time for a constant value of
A_∞^+. Naturally, this discussion assumes that the bubble
remains in the sound field at all times and is not subject to
any forces. In a free progressive-wave system this assump-
tion will often hold true. In a standing-wave system Bjerknes
forces will keep a bubble below resonant size at a pressure
antinode and a bubble above resonant size at a pressure node.
Because of these complications the growth curve of a gas-
filled bubble can be best obtained by means of numerical
integration of the nonlinear equations that describe the
growth of the bubble. These data have not been presented in
this memorandom, where the princip points of interest have
been (1) the minimum threshold for growth by means of recti-
fied diffusion, and (2) the characteristics of the pulsations
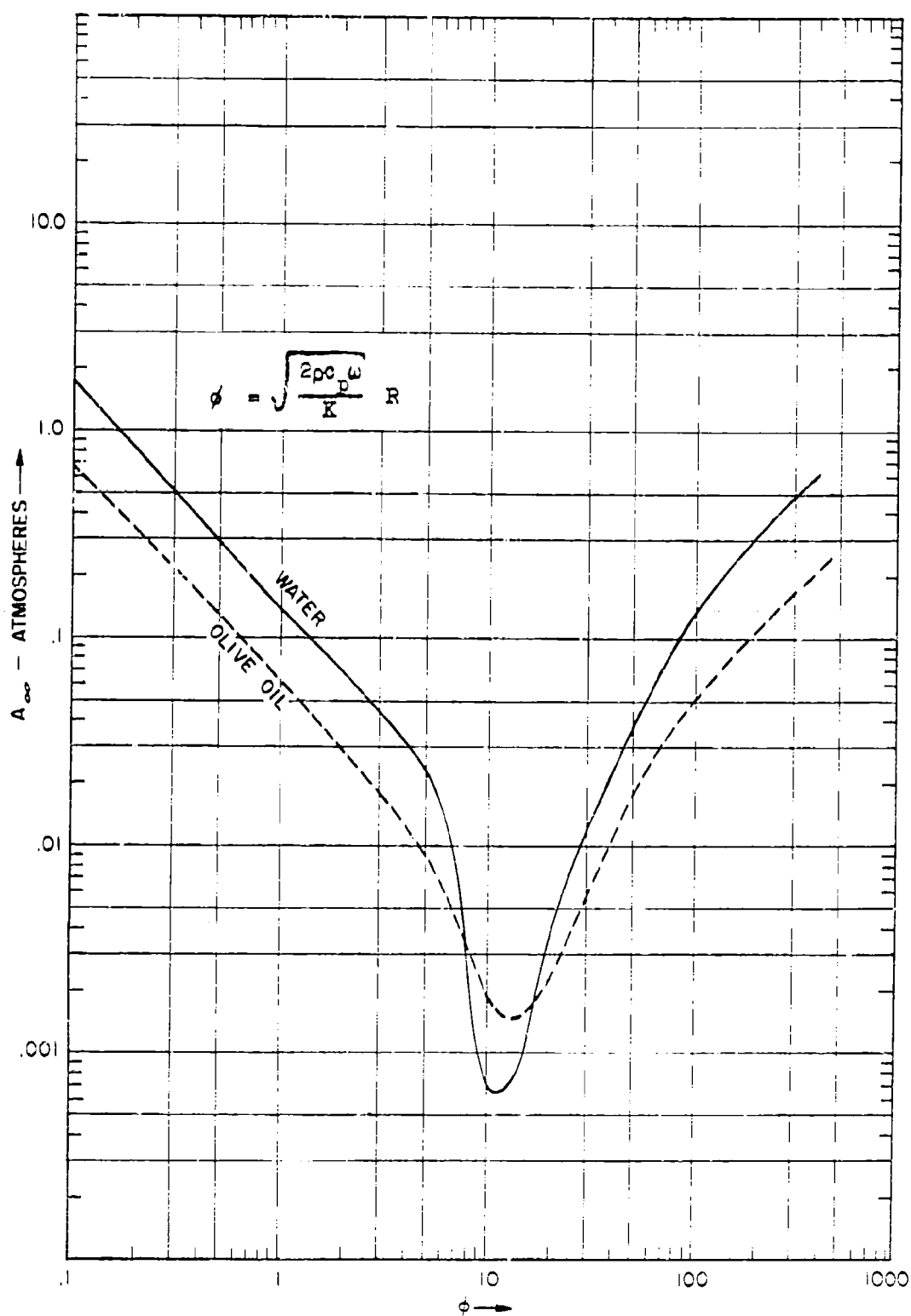of bubbles of various sizes.

$$\phi = \sqrt{\frac{2\rho c_p \omega}{K}}\ R$$

FIG. 4. THE THRESHOLD, A_∞, FOR RECTIFIED DIFFUSION

APPENDIX I

The Nonlinearity of Parabolic Equations
Subject to Moving Boundary Conditions

In this technical memorandum it is necessary to solve parabolic equations such as the heat conduction equation and the diffusion equation. For these equations the values of the temperature and gas concentration at the moving surface of the bubble can be taken as the boundary conditions. A parabolic equation subject to conditions at a moving boundary is nonlinear [18].

Consider the one-dimensional equation

$$\frac{\partial}{\partial x} F \frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} , \qquad (A\text{-}1)$$

where

$$f(x,0) = g(x) , \qquad R_0 < x < R', \qquad t = 0$$

$$\left. \frac{\partial f}{\partial x} \right)_{x=R'} = 0$$

$$\left. \begin{array}{c} \\ \\ \end{array} \right\} , \qquad x = R' \qquad t > 0$$

$$\text{or} \qquad f(R',t) = 0$$

$$f(R(t),t) = h(t) , \qquad x = R(t), \qquad t > 0$$

Let us introduce the transformation

$$x' = \frac{R(t) - x}{R(t) - R'} . \qquad (A\text{-}2)$$

Then

$$\frac{1}{(R(t)-R')^2} \frac{\partial}{\partial x'} \left(F \frac{\partial f}{\partial x'} \right) = \frac{\partial f}{\partial t} + \frac{1-x'}{R(t)-R'} \frac{dR(t)}{dt} \frac{\partial f}{\partial x'} , \quad 0 < x' < 1,$$

$$t > 0,$$

$$(A\text{-}3)$$

where

$$f(x,o) = g(x') \qquad 0 \leq x \leq 1, \qquad t = 0$$

$$\left. \frac{\partial f}{\partial x'} \right)_{x'=1} = 0$$

or $\qquad\qquad\qquad\qquad\qquad\qquad x' = 0 \quad , \qquad t > 0$

$$f(1, 0) = 0$$

$$f(R(t),t) = h(t) \qquad x' = 0 \quad , \qquad t > 0.$$

This equation is nonlinear unless $R(t)$ is constant. There-
fore, in studying the pulsations and growth of gas bubbles we
consider a spherical bubble immersed in an infinite liquid.
This bubble is in the path of a longitudinal sound wave.
In solving equations of the parabolic type we find it conven-
ient in many instances to assume that the radius of the bubble
is momentarily fixed. Later, we can introduce approximations
for the variations of bubble radius with time, namely, $R(t)$.

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